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ON THE SOLVABILITY

OF SOME BOUNDARY VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL PROBLEMS IN A FINITE INTERVAL

SOBRE LA SOLUBILIDAD DE ALGUNOS PROBLEMAS DEL VALOR LÍMITE PARA PROBLEMAS DIFERENCIALES POR EL OPERADOR EN UN INTERVALO FINITO

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ABSTRACT

In this paper, conditions are obtained for the coefficients of one class of operator-differential equations of the fourth order, which ensure the solvability of some boundary value problems correctly posed for these equations. Moreover, a connection is indicated between the solvability of boundary value problems and the exact value of the norm of operators of intermediate derivatives in some subspaces.

Keywords: Equations, solvability of boundary, mechanics, mathematical-physic.

RESUMEN

En este trabajo, se obtienen condiciones para los coeficientes de una clase de ecuaciones operador-diferenciales del cuarto orden, que aseguran la solvencia de algunos problemas de valor límite correctamente planteados para estas ecuaciones. Además, se indica una conexión entre la solvabilidad de los problemas de valor límite y el valor exacto de la norma de operadores de derivados intermedios en algunos subespacios.

Palabras clave: Ecuaciones, solvencia de límites, mecánica, física-matemática.

INTRODUCTION

Many problems in mechanics, mathematical physics, and the theory of partial differential equations lead to the study of the solvability of boundary value problems for operator-differential equations in various spaces (Kreyn & Laptev, 1966; Kreyn, 1967; Yurchuk, 1974; Gasimov & Mirzoev, 1992; Mirzoev & Humbataliev, 2010; Humbataliev, 2014ab, 2020, 2021).

Note that some problems in the theory of elasticity in a strip (Papkovich, 1940, 1941; Ustinov & Yudovich, 1973), problems in the theory of vibrations of mechanical systems (Kostyuchenko & Orazov, 1981), and vibrations of an elastic cylinder (Lyons & Magenes, 1971) lead to the study of the solvability of some boundary value problems for operator-differential equations and the construction of the spectral theory of quadratic beams and high-order beams. For example, the stress-strain state of a slab leads to solving problems of the theory of elasticity in a strip. In the works of Papkovich (1940, 1941; Ustinov & Yudovich (1973); and Orazov, 1979) the boundary value problem of elasticity theory in a strip reduces the solvability of various boundary value problems for a second-order equation and obtained a solution in the form of the limits of decreasing elementary solutions of the homogeneous equation, which is closely related to the double completeness of systems of eigenvectors and associated vectors.

Note that finding the exact values of the norms or their upper bounds for the operators of intermediate derivatives are of independent mathematical interest and have numerous applications in various fields of mathematical analysis (Lyons & Magenes, 1971; Gorbachuk & Gorbachuk, 1984; Mirzoev, 2003)1984; Lyons & Magenes, 1971; Mirzoev, 2003, for example, in approximation theory (Stechkin, 1967; Taikov, 1968).

In order to continue let's define some auxiliary facts and the problem statement. Let H be a separable Hilbert space, and A, a positive definite self-adjoint operator with domain D(A). Let denote by a scale Hilbert space generated by the operator A, i.e., $H_y = D(A^y)$, $(x,y)_y$, $x,y \in H_y$, $y \geq 0$, and when y = 0 then $H_0 = H$. We denote by $L_2((0,1); H)$ the Hilbert space of vector functions f(t), defined in (0,1) almost everywhere with values in H, such as equation 1 is fulfilled.

$$\|f\|_{L_2((0,1);H)} = \left(\int_0^1 \|f\|^2 dt\right)^{1/2} < \infty \qquad (1)$$

Further, define a Hilbert space (Lyons & Magenes, 1971) as equation 2 with a norm given in equation 3. Here and in what follows, derivatives are understood in the sense of the theory of distributions.

$$W_2^4((0,1);H) = \{u: u^{(4)} \in L_2((0,1);H), A^4u \in L_2((0,1);H)\} \qquad (2)$$

$$\|u\|_{W_2^4((0,1);H)} = \left(\left\|u^{(4)}\right\|_{L_2((0,1);H)} + \|A^4u\|_{L_2((0,1);H)}\right)^{1/2} \qquad (3)$$

In addition, let's introduce subspaces space shown in equation 4 where is a real number. Similarly let's define space $L_2(R;H)$ and $W_2^4(R;H)$ at $R=(-\infty,\infty)$. We denote $D((0,1); H_4)$ as the set of infinitely differentiable functions with values in H_4. As is known (Lyons & Magenes, 1971), the linear space $D((0,1); H_4)$ is dense everywhere in space $W_2^4((0,1); H_4)$. Then equation 5 follows from the trace theorem being $D((0,1); H_4)$ dense everywhere in space $W_2^4((0,1); H;a)$.

$$W_2^4((0,1);H):\alpha) = \{u: u \in W_2^4((0,1);H), u^{(k)}(0) = e^{i\alpha}u^{(k)}(1), k = \overline{0,3}\} \qquad (4)$$

$$D((0,1);H;\propto) = \{u: u \in D((0,1);H), u^{(k)}(0) = e^{i\alpha}u^{(k)}(1), k = \overline{0,3}\} \qquad (5)$$

In a separable Hilbert space, consider the boundary value problem of equations 6 and 7, where A = A ^ * >cE(c>0), but $A_j (j=\overline{0,3})$ is a linear operator in space.

$$P\left({}^{d}/_{dt}\right)u(t) = \left(\frac{d^2}{dt^2} - A^2\right)^2 u(t) + \sum_{j=1}^{4} A_j u^{(4-j)}(t) = f(t) \qquad (6)$$

$$u^{(k)}(0) = e^{i\alpha_k} u^{(k)}(1), k = \overline{0,3} \qquad (7)$$

Definition 1: If for f(t) ∈ L_2((0,1);H) there is a vector – function u(t) ∈ W_2^4((0,1); H), which satisfies equation (6) in almost everywhere, then it will be called a regular solution to equation (6).

Definition 2: If for any f(t) ∈ L_2((0,1);H) there is a regular solution to equation (6) u(t) ∈ W_2^4((0,1); H) which satisfies the boundary condition (7) in the sense of the convergence (equation 8) and the inequality (equation 9) then problem (6), (7) is called regularly solvable.

$$\lim_{t\to 0}\left\| u^{(k)}(t) - e^{i\alpha t} u^{(k)}(1-t) \right\|_{4-k-1/2} = 0, k = \overline{0,3} \qquad (8)$$

$$\|u\|_{W_2^4((0,1);H)} \le const\, \|f\|_{L_2((0,1);H)} \qquad (9)$$

Then, the objective of this paper is to find sufficient conditions on the coefficients of equation (6), which ensure the regular solvability of problem (6), (7). These conditions are expressed only by the coefficients of equation (6). Note that for equation (6) various boundary value problems in the semiaxis have been investigated by many authors (Kreyn & Laptev, 1966; Ustinov & Yudovich, 1973; Yurchuk, 1974; Gasimov & Mirzoev, 1992; Humbataliev, 2014ab, 2020, 2021).

DEVELOPMENT

In a finite domain, the existence and uniqueness of generalized solutions of some boundary value problems were studied by Orazov (1979); and Mirzoev & Humbataliev (2010), in a more general form. In Dubinsky (1973), the existence of periodic solutions to the boundary value problem was investigated, when the main part has the form of equation 10 but the coefficient expressed in equation 11 were complex numbers, and A-self-adjoint operators had a discrete spectrum. Note that for the boundary conditions are periodic, and for boundary conditions are antiperiodic. Also when n=1 the periodic problem is covered in the book by Kreyn (1967)"author":[{"family":"Kreyn","given":"S.G."}],"issued":{"date-parts":[["1967"]]},"suppress-author":true}],"schema":"https://github.com/citation-style-language/schema/raw/master/csl-citation.json"} .

$$(-1)^n \frac{d^{2n}}{dt^{2n}} + A^{2n} \qquad (10)$$

$$A_j (j = \overline{1,2n-1}) \qquad (11)$$

First, let's look at the simple problem of equations 12 and 13. Then, we will prove the lemma 1, which we will need in what follows.

$$P_0\left({}^{d}/_{dt}\right)u(t) = \left(\frac{d^2}{dt^2} - A^2\right)^2 u(t) = f(t) \qquad (12)$$

$$u^{(k)}(0) = e^{i\alpha_k} u^{(k)}(1), k = \overline{0,3} \quad (13)$$

Lemma 1: At u(t) ∈ $W_2^4((0,1); H;a)$ for any polynomial in the form of equation 14 where $\alpha_k \in R$ the inequality of equation 15 holds, where $b_0 = a_0^2$, $b_1 = a_1^2 - 2a_2a_0$, $b_2 = a_1^2 - 2a_3a_1 + 2a_4a_0$, $b_3 = a_3^2 - 2a_2a_4$, $b_4 = a_4^2$.

$$Q(\lambda; A) = \sum_{k=0}^{4} \alpha_k A^{4-k} \lambda^k \quad (14)$$

$$\left\| Q\left({}^{d}/_{dt} : A\right) u(t) \right\|^2_{L_2((0,1);H)} = \sum_{k=0}^{4} b_k \left\| A^{4-k} u^{(k)}(t) \right\|^2_{L_2((0,1);H)} \quad (15)$$

Proof: Obviously, it suffices to prove (15) for vector functions u(t) ∈ D((0,1);H;a). When u(t) ∈ D((0,1);H;a) equation 16 holds.

$$\begin{aligned} \left\| Q\left({}^{d}/_{dt} : A\right) u(t) \right\|^2_{L_2((0,1);H)} &= \sum_{k=0}^{4} a_k^2 \left\| A^{4-k} u^{(k)}(t) \right\|^2_{L_2(R_+;H)} \\ &+ \sum_{k=1}^{4} \sum_{s=0}^{k-1} a_k a_s 2Re\left(A^{4-k} u^{(k)}, A^{4-s} u^{(s)}\right)_{L_2((0,1);H)} \end{aligned} \quad (16)$$

When k=s+2v,v=1,2 integrating by parts, we obtain equation 17.

$$\begin{aligned} \left(A^{4-k} u^{(k)}, A^{4-s} u^{(s)}\right)_{L_2((0,1);H)} &= \left(A^{4-k-\frac{1}{2}} u^{(k-1)}(t), A^{4-s-\frac{1}{2}} u^{(s)}\right)\Big|_0^1 + \cdots + \\ (-1)^{\nu-1} &\left(A^{4-(s+\nu)-\frac{1}{2}} u^{(s+\nu)}(t), A^{4-(s+\nu-1)-\frac{1}{2}} u^{(s+\nu-1)}(t)\right)\Big|_0^1 \\ &+ (-1)^{\nu} \left\| A^{n-(s+\nu)} u^{(s+\nu)}(t) \right\|^2_{L_2(R_+;H)} \end{aligned} \quad (17)$$

As u(t) ∈ D((0,1);H;a), then we get equation 18. Similarly, we find equation 19 for s=v+1.

$$\begin{aligned} &\left(A^{4-k-\frac{1}{2}} u^{(k-1)}(1), A^{4-s-\frac{1}{2}} u^{(s)}(1)\right) - \left(A^{4-k-\frac{1}{2}} u^{(k-1)}(0), A^{4-s-\frac{1}{2}} u^{(s)}(0)\right) = \\ &\left(A^{4-k-\frac{1}{2}} u^{(k-1)}(1), A^{4-s-\frac{1}{2}} u^{(s)}(1)\right) - \left(e^{i\alpha} A^{4-s-\frac{1}{2}} u^{(k-1)}(1), e^{i\alpha} A^{4-s-\frac{1}{2}} u(1)\right) = 0 \end{aligned} \quad (18)$$

$$Re\left(A^{4-k}u^{(k)}, A^{4-s}u^{(s)}\right)_{L_2((0,1);H)} = -Re\left(A^{4-k}u^{(k)}, A^{4-s}u^{(s)}\right)_{L_2((0,1);H)} \text{, i.e. } 2Re\left(A^{4-k}u^{(k)}, A^{4-s}u^{(s)}\right)_{L_2((0,1);H)} = 0 \quad (19)$$

In this way we can get equation 20 and the lemma 1 is proved.

$$\left\|Q\left({}^{d}/_{dt} : A\right)u(t)\right\|^2_{L_2((0,1);H)} = \sum_{k=0}^{4} a_k^2 \left\|A^{4-k}u^{(k)}(t)\right\|^2_{L_2(R_+;H)} + \sum_{k=1}^{4}\sum_{s=0}^{l-1} a_k a_s \left\|A^{4-\left(\frac{k+s}{2}\right)}u^{\left(\frac{k+s}{2}\right)}\right\|_{L_2} (-1)^{\left(\frac{k-s}{2}\right)} = \sum_{k=0}^{4} b_k \left\|A^{4-k}u^{(k)}(t)\right\|^2_{L_2((0,1);H)} \quad (20)$$

Corollary 1: At $u(t)\in W_2^4((0,1);H;\alpha)$ the equality in equation 21 holds.

$$\left\|P_0\left({}^{d}/_{dt}\right)u\right\|^2_{L_2(R_+:H)} = \left\|u^{(4)}\right\|^2_{L_2((0,1);H)} + 4\left\|Au'''\right\|^2_{L_2((0,1);H)} + 6\left\|A^2u''\right\|^2_{L_2((0,1);H)} + 4\left\|A^3u'\right\|^2_{L_2((0,1);H)} + \left\|A^4u\right\|^2_{L_2((0,1);H)} \quad (21)$$

Indeed, from the previous expression it's clear that $a_0 = 1$, $a_4 = 1$, $a_2 = -2$, $a_1 = a_1 = 0$. Therefore, it follows from Lemma 1 that $b_0 = 1$, $b_4 = 1$, $b_1 = 4$, $b_2 = 6$, $b_3 = 4$. Then, the following theorem takes place.

Theorem 1: Operator P_0 maps space isomorphically $W_2^4((0,1);H;\alpha)$ still $L_2((0,1);H)$.

Proof: From Corollary 1, it follows that Ker $P_0=\{0\}$. Let's prove that its range of value Jm $P_0=L_2((0,1);H)$. To this end, let's prove that problem of equations 12 and 13 has a unique regular solution. For a given $f(t)\in L_2((0,1);H)$ construct a vector function in the form of equation 22 and show that $u_1(t) \in W_2^4(R,H)$. By Plancherel's theorem, it suffices to prove the conditions of equation 23.

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\xi^2E + A^2)^{-2} \int_0^1 f(s)e^{i(t-s)}dsd\xi, t \in R \quad (22)$$

$$A^4\hat{u}_1(\xi) \in L_2 \text{ and } \xi^4\hat{u}(\xi) \in L_2(R,H)\left(R = (-\infty, +\infty)\right) \quad (23)$$

It's obvious the relation of equation 24 where $\hat{f}_1(\xi)$ is the Fourier transform of the vector function $f_1(t)$, which is a continuation of the function f(t) from interval (0,1) on R as a null function. Then equation 25 holds.

$$\hat{u}_1(\xi) = (\xi^2 E + A^2)^{-2}\hat{f}_1(\xi) \quad (24)$$

$$\|A^4 u_1(\xi)\| = \left\|A^4(\xi^2 E + A^2)^{-2}\hat{f}_1\right\|_{L_2(R,H)} \leq \sup_{\xi\in R}\|A^4(\xi^2 E + A^2)\|^{-2} \cdot \|f\|_{L_2(R,H)} \quad (25)$$

Since at ξ∈R we get the relation of equation 26 arriving at equation 27 and similarly .

$$\left\|A^4(\xi^2 E + A^2)^{-2}\hat{f}_1\right\|_{L_2(R,H)} \leq \sup_{\xi\in R}|\mu^4(E^2+\mu)|^{-2} \leq 1 \quad (26)$$

$$A^4 u_1(\xi) \in L_2(R_+, H) \quad (27)$$

We denote the narrowing $u_1(\xi)\in W_2^2(R, H)$ on (0,1) through $\xi_1(t)$. Then it is obvious that and we get the equation 28.

$$\xi_1{}^{(k)}(0) \in H_{4-k-\frac{1}{2}}, \xi_1{}^{(k)}(1) \in H_{4-k-\frac{1}{2}}, k = \overline{0,3} \quad (28)$$

Now let´s look for regular solutions problem (12), (13) in the form of equation 29 where c_0, c_1, c_2, c_3 are unknown vectors from space $H_{7/2}$ (Yurchuk, 1974) and belong to the definition. To define vectors c_0, c_1, c_2, c_3 we get the following system of equations (equation 30).

$$u(t) = \xi_1(t) + e^{-tA}c_0 + tAe^{-tA}c_1 + e^{(t-1)A}c_2 + (t-1)Ae^{(t-1)A}c_3 \quad (29)$$

$$\begin{aligned}(E - e^{i\alpha}e^{-A})c_0 + e^{i\alpha}Ae^{-A}c_1 - (e^{-A} - e^{i\alpha}E)c_3 - Ae^{-A}c_3 = e^{i\alpha}\big(\xi(1)\\ (-E + e^{i\alpha}e^{-A})c_0 + (E + e^{i\alpha}E - A)e^{-A}c_1 + (e^{-A} - e^{i\alpha})c_2 - (e^{-A} - e^{i\alpha})c_3 = e^{i}\\ (E - e^{i\alpha}e^{-A})c_0 + (-2E - (E-A)e^{i\alpha})e^{-A}c_1 + (e^{-A} - e^{i\alpha})c_2 + (e^{-A}(2-A) - 2e^{i\alpha})c_3\\ (-E + e^{i\alpha}e^{-A})c_0 + (-3E - (3A^3 - A^4)e^{i\alpha})e^{-A}c_1 + (e^{-A} - e^{i\alpha})c_2 + (e^{-A}(3E - A) - 3e^{i\alpha})\end{aligned} \quad (30)$$

This system can also be represented in operator form $W(A)\tilde{c}=\tilde{\psi}$ where $\tilde{c}$ = c_0, c_1, c_2, c_3, $\tilde{\psi}=(\psi_0,\psi_1,\psi_2,\psi_3)$ and . It's obvious that $\psi_k\in H_{7/2}$, $k=\overline{0,3}$. Let us show that the operator matrix reversible in space $H_{7/2}$.

Let $\lambda\in[\mu_0,\infty)$ and denote W(λ), where in the matrix W(A) operator A replaced through $\lambda\in[\mu_0,\infty)$. Obviously, for λ→∞ the relation of equation 31 is fulfilled.

$$\det W(\lambda) = \begin{vmatrix} 1 & 0 & -e^{i\alpha} & 0 \\ -1 & 1 & -e^{i\alpha} & -e^{i\alpha} \\ 1 & -2 & -e^{i\alpha} & -2e^{i\alpha} \\ -1 & 3 & -e^{i\alpha} & -3e^{i\alpha} \end{vmatrix} + 0(\lambda) = 4e^{2i\alpha} + 0(\lambda) \quad (31)$$

Thus, for $\lambda>\Lambda_0$, where Λ_0 is quite a large number, det W(λ) is reversible: $|\det W^{-1}(\lambda)| \leq$ const. Let $\lambda \in [\mu_0, \Lambda_0]$, then for $\forall\lambda \in [\mu_0, \Lambda_0]$ W(λ) is reversible and $|\det W^{-1}(\lambda)| \leq$ const. If for some $\forall\lambda \in [\mu_0, \Lambda_0]$ $W(\lambda_0)$ is not reversible, then det W(λ) = 0, i.e., $W(A)\tilde{\chi}=0$, $\tilde{\chi}=(\chi_1,\chi_2,\chi_3,\chi_4)$, has a nonzero solution. This means that the boundary value problem of equation 32 has a nonzero solution, and when H=C, A=λ0, which contradicts corollary 1.

$$\begin{aligned}&\left(\frac{d^2}{dt^2} - A^2\right)^2\varphi(t) = 0\\ &\varphi^{(k)}(0) = e^{i\alpha k}\varphi^{(k)}(1), k = \overline{0,3}\end{aligned} \quad (32)$$

Thus, for any $\lambda\in[\mu_0, \Lambda_0]$ exists $W^{-1}(\lambda_0)$. On the other hand $W^{-1}(\lambda)$ is a continuous function, then $\|W^{-1}(\lambda)\|\leq$const in $[\mu_0, \Lambda_0]$. But with λ→∞ $\|W^{-1}(\lambda)\|\leq$const. Thus, the inequality $\|W^{-1}(\lambda)\|\leq$const is also carried out at $\lambda\in\sigma(A)\in[\mu_0,\infty)$. Then from the spectral decomposition A follows that W(A) is reversible in $H_{7/2}$ and $\|W^{-1}(\lambda)\|\leq$const. Therefore, we can find c_0, c_1, c_2 and

$c_3 \epsilon H_{7/2}$. Thus, in u(t) there is a solution $P_0u=f$. On the other hand, equation 33 follows from Lemma 1 of the theorem on intermediate derivatives.

$$\|P_0 u\|^2_{L_2((0,1);H)} = \left\|P_0\left({}^{d}/_{dt}\right)\right\|^2_{L_2((0,1);H)} \le const\ \|u\|^2_{W_4^2((0,1);H)} \qquad (33)$$

Then the theorem on the inverse operator implies the assertion of Theorem 1. Thus, the expressions in equation 34 are equivalent in space $W_4^2((0,1);H)$.

$$\|u\|^2_{W_4^2((0,1);H)}$$
$$\left\|P_0\left({}^{d}/_{dt}\right)\right\|^2_{L_2((0,1);H)} \qquad (34)$$

Then it follows from the theorem on intermediate derivatives that the following numbers of equation 35 are finite.

$$N_k = \sup_{0 \ne u \in W_4^2((0,1);H)} \|A^{4-k}\|\ \|P_0 u\|^{-1}, k = \overline{0,3} \qquad (35)$$

On the solvability of boundary value problems (6), (7)

Taking the above into account the theorem 2 holds.

Theorem 2: Let A=A*≥cEFc>0) and the operators $B_j=A_jA^{-j}$Fj=0,4) bounded in H. Then, when the inequality α= ù$^4_{j=0}$ N_j B_{4-j}‖<1, where N_jFj=0,3) are defined from equality (35), then problem (6), (7) is regularly solvable.

Proof: Let's write problem (6), (7) in the form of the equation 36 where fFt)∈L_2FF0,1);H), uFt)∈W_4^2FF0,1);H). The operator P_0:W_4^2FF0,1);H)→L_2FF0,1);H) is isomorphic,| then for any vFt)∈L_2FF0,1);H) there will be uFt)∈W_4^2(F0,1);H;α), such that vFt)=P_0^{-1}uFt). Then we obtain the equation 37.

$$Pu(t) = P_0 u(t) + P_1 u(t) = f(t) \qquad (36)$$

$$v(t) + P_1 P_0^{-1} v(t) = f(t) \qquad (37)$$

The relation in equation 38 is fulfilled considering that α<1, and we get that v(t) = (E+ $P_1P_0^{-1}$) f(t), but u(t)=P_0 (E+ P_1 P_0^{-1}) f(t).

$$\|P_1 P_0^{-1} v(t)\|_{L_2((0,1);H)} = \|P_1 u\|_{L_2((0,1);H)} = \left\|\sum_{j=0}^{3} A_{4-j} u^{(j)}\right\|_{L_2((0,1);H)}$$
$$\le \left\|\sum_{j=0}^{3} B_{4-j}\right\| \|A^{4-j}u^{(j)}\|_{L_2((0,1);H)} \le \sum_{j=0}^{3} N_j \|B_{4-j}\| \|P_0 u\|_{L_2((0,1);H)} \qquad (38)$$
$$= \sum_{j=0}^{3} N_j \|B_{4-j}\| \|v\|_{L_2((0,1);H)} = \alpha \|v\|_{L_2((0,1);H)}$$

Therefore, equation 39 follows and the theorem is proved.

$$\|u\|_{W_4^2((0,1);H)} \leq \left\| P_0 \, {}^{d}/_{dt} \right\|_{L_2((0,1);H)} \tag{39}$$

Thus, to find the condition for regular solvability, we must find upper bounds for the norms $N_j=(j=\overline{0,3})$. For this purpose, consider the operator P_j (λ,β,A) = $(-\lambda^2 + A^2)^4 - (-1)^j \beta\lambda^{2j} A^{8-2j}$ $(j=\overline{0,3})$ where β is a valid parameter.

Lemma 2: Let $\beta \in (0, d_{4,j}^{-4})$, where $d_{4,j}^{-4})$ is expressed in the form of equation 40. Then the operator P_j (λ,β,A) is reversible on the imaginary axis and is represented in the form of equation 41 where the beam coefficient may be expressed as equation 42 and every $\alpha_{j,l}$ (β)>0 satisfy the relation as can be seen next.

$$d_{4,j}^{-4} = \left(\frac{j}{4}\right)^{\frac{j}{4}} \left(\frac{4-j}{4}\right)^{\frac{4-j}{4}}, j = 1,2,3 \tag{40}$$

$$P_j(\lambda, \beta, A) = F_j(\lambda, \beta, A) F_j(-\lambda, \beta, A) \tag{41}$$

$$F_j(\lambda, \beta, A) = \sum_{l=0}^{4} \alpha_{j,l} \lambda^j A^{4-j} \tag{42}$$

a) At ***j=1***, equation 43.

$$\alpha_{1,0} = \alpha_{1,4} = 1$$
$$\begin{cases} \alpha_{1,1}^2(\beta) - 2\alpha_{1,2}(\beta) = 4 - \beta \\ \alpha_{1,2}^2(\beta) - 2\alpha_{1,1}(\beta)\alpha_{1,3}(\beta) = 4 \\ \alpha_{1,3}^2(\beta) - 2\alpha_{1,2}(\beta) = 4 \end{cases} \tag{43}$$

b) At , equation 44.

$$\alpha_{2,0} = \alpha_{2,4} = 1$$
$$\begin{cases} \alpha_{2,1}^2(\beta) - 2\alpha_{2,2}(\beta) = 4 \\ \alpha_{2,2}^2(\beta) - 2\alpha_{2,1}(\beta)\alpha_{2,3}(\beta) = 4 - \beta \\ \alpha_{2,3}^2(\beta) - 2\alpha_{2,2}(\beta) = 4 \end{cases} \tag{44}$$

c) At , equation 45.

$$\alpha_{3,0} = \alpha_{3,4} = 1$$
$$\begin{cases} \alpha_{3,1}^2(\beta) - 2\alpha_{3,2}(\beta) = 4 \\ \alpha_{3,2}^2(\beta) - 2\alpha_{3,1}(\beta)\alpha_{3,3}(\beta) = 4 \\ \alpha_{3,3}^2(\beta) - 2\alpha_{3,2}(\beta) = 4 - \beta \end{cases} \tag{45}$$

Proof: Let σ∈σ(A), λ=iξ, ξ∈R. Then it is obvious that for the numerical polynomials of equation 46 i.e., at σ∈σ(A) polynomial are on the imaginary axis.

$$P_j(i\xi;\beta;\sigma) = (\xi^2+\sigma^2)^4 - (-1)^j \beta \xi^{2j} \sigma^{8-2j} = (\xi^2+\sigma^2)^4 (1 - \beta \xi^{2j} (\xi^2+\sigma^2)^{-4}) \geq (\xi^2+\sigma^2)^4 \left[1 - \beta \sup_{\xi/\sigma} \frac{\left(\xi/\sigma\right)^j}{\left(\xi^2/\sigma^2 + 1\right)^4} \right] = (\xi^2+\sigma^2)^4 \left(1 - \beta d_{4,j}^4\right) > 0 \tag{46}$$

Then the polynomial P_j $(i\xi;\beta;\sigma)$ has four roots from the left half-plane $\lambda_{l,j}(\beta;\sigma)=\omega_{l,j}(\beta)\sigma$, $l=\overline{1,4}$, $Re\ \omega_{l,j}<0$, $j=\overline{1,4}$ and four roots from the left half-plane $\lambda_{l,j}(\beta;\sigma)=-\omega_{l,j}(\beta)\sigma$, $l=\overline{1,4}$, $j=\overline{1,4}$. Hence:

$$P_j(\lambda;\beta;\sigma)=\prod_{l=1}^{4}\left(\lambda-\omega_{l,j}(\beta)\sigma\right)\prod_{l=1}^{4}\left(\lambda+\omega_{l,j}(\beta)\sigma\right) \quad (47)$$

Given that

$$F_j(\lambda;\beta;\sigma)=\prod_{l=1}^{4}\left(\lambda-\omega_{l,j}(\beta)\sigma\right)\prod_{l=0}^{4}\left(\alpha_{l,j}\lambda^{l}\sigma^{4-l}\right) \quad (48)$$

We get

$$P_j(\lambda;\beta;\sigma)=F_j(\lambda;\beta;\sigma)F_j(-\lambda;\beta;\sigma) \quad (49)$$

Further, using the spectral decomposition of the operator we obtain relation (41). Obviously every α_l,4(β)=1. But, $\alpha_{0,l}(\beta)=\prod_{l=1}^{4}\omega_{l,j}(\beta)$. Since the coefficients of the polynomial P_j $(\lambda;\beta;\sigma)$ are valid then roots $\omega_{l,j}$ (β) are complex conjugate or negative numbers. Therefore $\alpha_{0,j}$ (β)>0. On the other hand, it follows from (48) and (49) that $\alpha_{0,j}^2(\beta)=1$, i.,e., $\alpha_{0,j}$ (β) = 0. It follows from (49) that the remaining numbers . When comparing degrees equality (49) yields the validity of equality (43), (44), and (45) and the lemma is proved.

Corollary 2: With all $u(t)\in W_4^2\left((0,1);H;\alpha\right)$ the inequality of equation 50 holds.

$$\left\|F_j\left({}^{d}/_{dt}:\beta:A\right)u\right\|_{L_2((0,1);H)}^2=\|P_0u\|_{L_2((0,1);H)}^2-\beta\left\|A^{4-j}u^{(j)}\right\|_{L_2((0,1);H)}^2 \quad (50)$$

Proof: As equation 51 is fulfilled then applying relations (43) - (45) and using Corollary 1, we obtain the relation in equation 52.

$$\left\|F_j\left({}^{d}/_{dt}:\beta:A\right)u\right\|_{L_2((0,1);H)}^2=\left\|\sum_{k=0}\alpha_{k,j}(\beta)A^{4-k}\,u^{(k)}\right\|_{L_2((0,1);H)}^2 \quad (51)$$

$$\begin{aligned}\left\|F_j\left({}^{d}/_{dt}:\beta:A\right)u\right\|_{L_2((0,1);H)}^2&=\left\|u^{(4)}\right\|_{L_2((0,1);H)}^2+4\left\|Au^{(3)}\right\|_{L_2((0,1);H)}^2+6\|A^2u''\|_{L_2((0,1);H)}^2+\\ &4\|A^3u'\|_{L_2((0,1);H)}^2+\|A^4u\|_{L_2((0,1);H)}^2-\beta\left\|A^{4-j}u^{(j)}\right\|_{L_2((0,1);H)}^2\\ &=\|P_0u\|_{L_2((0,1);H)}^2-\beta\left\|A^{4-j}u^{(j)}\right\|_{L_2((0,1);H)}^2\end{aligned} \quad (52)$$

Theorem 3: With all $u(t)\in W_4^2((0,1);H;\alpha)$ the inequality of equation 53 holds. Going to the limit $\beta\to d_{4,j}^{-4}$ we get the relation in equation 54 and the theorem is proved.

$$\left\|A^{4-j}u^{(j)}\right\|_{L_2((0,1);H)}^2\le\|P_0u\|_{L_2((0,1);H)}^2\ ,j=1,2,3 \quad (53)$$

$$\begin{aligned}&\|A^{4-j}u^{(j)}\|_{L_2((0,1);H)}^2\le d_{4,j}^{-4}\|P_0u\|_{L_2((0,1);H)}^2\ ,j=1,2,3\\ &\textit{i.e. } N_j(\alpha)\le d_{4,j}^{-4}\ ,j=1,2,3\end{aligned} \quad (54)$$

Then, from Theorems 2 and 3 we obtain the following main theorem

Theorem 4: Let $A=A^*\geq cE$ $(c>0)$, and the operators $B_j=A_jA^{-j}$ $(j=\overline{0,4})$ bounded in , if the inequality $\alpha=\sum_{j=0}^{4} d_{4,j}^{-2}\|B_{4-j}\|<1$ holds, then problem (6), (7) is regularly solvable.

CONCLUSIONS

The theory of partial differential equations lead to the study of the solvability of boundary value problems for operator-differential equations in various spaces. In this work, applying logic, sufficient conditions were found for the solvability of a special type of boundary value problem, which were synthesized in theorem 4. This has special relevance for solving problems in different fields, especially mathematical physics.

REFERENCES

Dubinsky, Y. A. (1973). Periodic solutions of elliptic-parabolic equations. *Mathematical Collection*, *90(132)* (1), 3–22.

Gasimov, M. G., & Mirzoev, S. S. (1992). On the solvability of boundary value problems for second-order operator-differential equations of elliptic type. *Differential Equations*, *28*(4), 651–661.

Gorbachuk, V. I., & Gorbachuk, M. L. (1984). *Boundary value problems for differential-operator equations*. Naukova Dumka.

Humbataliev, R. Z. (2014a). On solvability of some boundary value problems for elliptic type operator-differential equations. *Eurasian Mathematical Journal*, *5*(4), 33–55.

Humbataliev, R. Z. (2014b). *On the solvability of boundary value problems for operator-differential equations and some spectral problems*. Nauka.

Humbataliev, R. Z. (2020). Completeness and minimality of a system of generalized decreasing elementary solutions. *Fundamental Problems of Mathematics and the Application of Intellectual Technologies in Education*, 7–9.

Humbataliev, R. Z. (2021). On the periodicity of a boundary value problem. *Scientific Horizon in the Context of Social Crises*, *50*, 567–574.

Kostyuchenko, A. G., & Orazov, M. B. (1981). The problem of oscillation of an elastic half-cylinder and related quadratic beams. *Proceedings of the I. G. Petrovsky Seminar*, *6*, 97–146.

Kreyn, S. G. (1967). *Linear differential equations in a Banach space*. Nauka.

Kreyn, S. G., & Laptev, G. I. (1966). Correctness of boundary value problems for differential equations in a Banach space. *Differential Equations*, *2*(3), 382–390.

Lyons, L. J., & Magenes, E. I. (1971). *Homogeneous boundary value problems and their applications*. MIR.

Mirzoev, S. S. (2003). On the norms of operators of intermediate derivatives. *Izv. NAN Azerb.*, *23*(1), 157–164.

Mirzoev, S. S., & Humbataliev, R. Z. (2010). On the completeness of the system of elementary solutions of one class of an operator-differential equation on a finite segment. *Doklady RAN*, *431*(4), 454–456.

Orazov, M. B. (1979). Completeness of elementary solutions of some operator differential equations on the semiaxis and segment. *Proceedings of the USSR Academy of Sciences (DAN SSSR)*, *245*(4), 788–792.

Papkovich, P. F. (1940). On one of the forms of solving the plane problem of the theory of elasticity for a rectangular strip. *Proceedings of the USSR Academy of Sciences (DAN SSSR)*, *37*(4).

Papkovich, P. F. (1941). Two questions in the theory of bending of thin elastic plates. *Journal of Applied Mathematics and Mechanics (PMM)*, *5*(3), 359–374.

Stechkin, . B. (1967). Best approximations of linear operators. *Mathematical Notes*, *1*(2), 137–138.

Taikov, L. V. (1968). Kolmogorov-type inequalities and best formulas for numerical differentiation. *Mathematical Notes*, *4*(2), 233–238.

Ustinov, Y. A., & Yudovich, Y. I. (1973). On the completeness of systems of elementary solutions of the biharmonic equation in a half-strip. *Journal of Applied Mathematics and Mechanics (PMM)*, *37*(4), 706–714.

Yurchuk, N. I. (1974). On boundary value problems for equations containing in the main part operators in the form. *Differential equations*, *10*(4), 759–762.