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METHODOLOGICAL

APPROACHES TO DEAL WITH UNCERTAINTY IN DECISION MAKING PROCESSES

ENFOQUES METODOLÓGICOS PARA EL TRATAMIENTO DE LA INCERTIDUMBRE EN LOS PROCESOS DE DECISIÓN

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ABSTRACT

The objective of this investigation is to discuss qualitatively the different methodological approaches developed to deal with uncertainty in decision making processes. For its preparation were used mainly the analysis of documents, the historical-logical method and the analytical-synthetic method which allowed an assessment of the state of the art in the topic. It was possible to identify that the phenomenon of uncertainty has two natures: one aleatory and other epistemic. Aleatory uncertainty arises from stochastic processes, while epistemic uncertainty is caused by imprecision, ignorance, credibility or incompleteness in the information necessary to make the decision. Aleatory uncertainty is effectively modeled by probability theory, which constitutes the starting point for maximizing expected utility in decision processes. Epistemic uncertainty is modeled, depending on the characteristic of the information, mainly through fuzzy sets theory, rough sets or gray systems. Each of these approaches has its advantages and disadvantages, so in order to take advantage of their strengths, hybrid models have been created. Nowadays, given the need to make more robust decisions, all these theories are being refined by the scientific community because, although uncertainty cannot be completely eliminated they have shown that it can be dealt with effectively.

Keywords: Uncertainty, decision making processes, methodological approaches.

RESUMEN

El objetivo de esta investigación consiste en discutir cualitativamente los diferentes enfoques metodológicos desarrollados para el tratamiento de la incertidumbre en los procesos de decisión. Para su elaboración se empleó fundamentalmente los métodos de análisis de documentos, histórico-lógico y analítico sintético lo que permitió una valoración del estado del arte en la temática. Se pudo identificar que el fenómeno de la incertidumbre tiene dos naturalezas: una aleatoria y otra epistémica. La incertidumbre aleatoria surge de procesos estocásticos mientras que la epistémica es provocada por imprecisión, ignorancia, credibilidad o incompletitud en la información necesaria para tomar la decisión. La incertidumbre aleatoria es modelada efectivamente mediante la teoría de las probabilidades, la que constituye el punto de partida para en los procesos de decisión maximizar la utilidad esperada. La incertidumbre epistémica es modelada, en dependencia de las características de la información, fundamentalmente mediante la teoría de conjuntos difusos, los conjuntos ásperos o los sistemas grises. Cada uno de estos enfoques tiene sus ventajas y desventajas por lo que con el propósito de aprovechar sus respectivas fortalezas se han creado modelos híbridos. Dada la necesidad de tomar decisiones más robustas en el presente todas estas teorías están siendo refinadas por la comunidad científica pues, aunque no se puede eliminar completamente la incertidumbre han demostrado que sí se puede lidiar con ella efectivamente.

Palabras clave: Incertidumbre, procesos de decisión, enfoques metodológicos.

INTRODUCTION

According to Peterson (2009), decision theory is a multidisciplinary project in which philosophers, economists, psychologists, computer scientists and statisticians contribute from their respective specialties. However, theorists from all disciplines share a number of basic concepts and distinctions, being fundamental the consensus on the distinction between descriptive decision theory and normative decision theory. Descriptive decision theory seeks to explain and predict how people actually make decisions (this is an empirical discipline that comes from experimental psychology) while normative theory seeks to reveal prescriptions about how decision makers, in a rational way, should proceed in a given situation. Thus these are presented as two separate fields that must be studied independently. Therefore, although the understanding of people's behavior in the face of decision-making is a very interesting aspect, for practical purposes there is greater interest in the theory of normative decision.

It is not known exactly when the formal study of this field began, but an important milestone can be identified in the development of utility theory by von Neumann and Morgenstern. Other significant advances took place during World War II with the birth of operations research (OR) when, according to Taha (2017), british scientists analyzed decisions regarding the best ways to use war material. At the end of the war, the success of the OR in war activities generated great interest due to the possibilities of applying it in a field other than the military one. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations were again coming to the forefront and then it began clear that these problems were essentially the same as those faced by the military but in a different context (Hillier & Lieberman, 2015).

However, with increasing frequency, traditional OR techniques are not fully adequate for decision-making due to the deep-rooted uncertainty of the contemporary context. According to Walker, Lempert & Kwakkel (2013), in a broad sense, uncertainty can be defined simply as a limitation in the knowledge of current, past or future events. Regarding decision-making processes, uncertainty refers to a gap between the available knowledge and the knowledge needed by decision-makers to implement the best policies. This uncertainty clearly involves subjectivity since satisfaction is related with existing knowledge, which is colored by the underlying values and perspectives of the decision maker (and of the various actors involved in the decision-making process). However, this in itself becomes a trap when implicit assumptions are left unexamined or unquestioned. Therefore, uncertainty itself can be associated with all aspects of a problem of interest (for example: the system that comprises the decision domain, the world outside the system, and the importance that stakeholders give to the various results of the system) (Marchau, et al., 2019).

Chen & Hwang (1992), point out that from a philosophical point of view, uncertainty is given by: (a) unquantifiable information, (b) incomplete information, (c) information impossible to obtain and /or (d) partial ignorance. Until the 20th century the preferred theory for modeling uncertainty was probability theory, but the introduction of fuzzy sets by Zadeh (1965), had a profound impact on the notion of uncertainty as well as on classical binary logic. Parallel to the theory of fuzzy sets, other theories have been developed such as rough sets theory, gray systems as well as extensions of fuzzy logic such as intuitionistic fuzzy logic or, more recently, neutrosophy. In addition, these theories have been combined to create hybrid models, trying to take advantage of the individual strengths of each one to treat uncertainty in the decision models more effectively.

Then, due to the continuous need to deal with decision processes with high uncertainty, the following article aims to discuss the various theories for their treatment from a qualitative point of view. For this, the main methods used were analysis of documents, the historical-logical method and the analytical-synthetic method.

DEVELOPMENT

Klir & Yuan (1995), point out that several classes of decision-making problems are usually recognized. According to one criterion, decision problems are classified as those involving a single decision maker and those which involve several decision makers. These problem classes are referred to as individual decision making and multiperson decision making, respectively. According to another criterion, we distinguish decision problems that involve a simple optimization of a utility function, an optimization under constraints, or an optimization under multiple objective criteria. Furthermore, decision making can be done in one stage, or it can be done iteratively, in several stages.

In his classic study, the decision analysis is expressed in the form of a problem where the decision-maker must select from a set of possible alternatives the one with the best performance according to some rational criterion. After the decision, factors or external uncontrollable variables (known as states of nature) will act to determine the outcome of the decision. An underlying assumption is that if the outcome of states of nature could be accurately predicted, then the end result would also be predictable and the correct alternative would become obvious (Carter, Price & Rabadi, 2019). This means that every decision, even if it is presumably correct, has an element of risk associated with it that can be valued and reduced within limits, but never completely eliminated since it is impossible to predict the future. Sometimes the risk is lower because these are problems that the decision-maker has faced previously, so he tends to adopt the same behavior experienced successfully on previous occasions. However, the more complex the problem and the more alternatives present, as well as the number and importance of the variables in the decision, the higher the risk coefficient in making the decision.

Bacci & Chiandotto (2020), emphasize that a relevant aspect that the decision maker must consider in this context is the level of knowledge about the states of nature. In detail, decisions can be distinguished according to the informational background in which the decision maker operates, which can be: 1) decisions in situations of certainty when states of nature are known 2) decisions in risk situations, when states of nature are unknown, but the decision maker has or can estimate a probability distribution for the set of states of nature and 3) decisions in situations of uncertainty, when the decision maker is unable or unwilling to proceed with the measurement of the plausibility of states of nature. Based on this, decision models have been developed which depend on several factors such as the degree of definition of the alternatives and the attributes used in the evaluation, the availability of data for the construction of the models, the availability of time to take the decision, as well as the repercussions or importance of it.

Considering the phenomenon of uncertainty, at the precise moment of making a decision the decision-maker is faced with two fundamental types, one of a random nature and the other of epistemic nature (Figure 1). Random uncertainty is the uncertainty that deals with the variability inherent in the physical world. Variability is often attributed to a random process that produces the natural variability of a quantity over time and space or between members of a population. It is, in principle, irreducible. In other words, the variability cannot be altered by obtaining more information, although one's characterization of that variability could change given the additional information. Random uncertainty is sometimes called variability, irreducible uncertainty, stochastic uncertainty, and random uncertainty. On the other hand, epistemic uncertainty is reducible in principle, although it may be difficult or expensive to do so. This arises from an incomplete theory and an incomplete understanding of a system, modeling limitations, or limited data. Epistemic uncertainty has also been called internal, functional, subjective, reducible, or model uncertainty. Knowledge uncertainty is easy to remember and perhaps a more descriptive term to describe this type of uncertainty (Yoe, 2019).

As can be seen in Figure 1 four factors can lead to epistemic uncertainty. Imprecision corresponds to the inability to express the true value because the absence of experimental values does not allow the definition of a probability distribution or because it is difficult to obtain the exact value of a measure. For instance, only bounds are known because it cannot be different physically. Ignorance (partial or total) corresponds to the inability to express knowledge on disjoint hypotheses. Sometimes, it is easier to express knowledge on their disjunctions. Indeed, what is more imprecise is more certain. Incompleteness corresponds to the fact that not all situations are covered. For instance, all the failure modes of a material are not known. Credibility concerns the weight that an agent can attach to its judgment. It is a sort of second-order information. Imprecision, ignorance and incompleteness are closed notions. However, incompleteness is a kind of model uncertainty, whereas ignorance and imprecision more concern parametric uncertainty. Also, imprecision and ignorance are different because the first is linked to the quality of the value, whereas the second is associated with the knowledge of the value (Simon, et al., 2018).

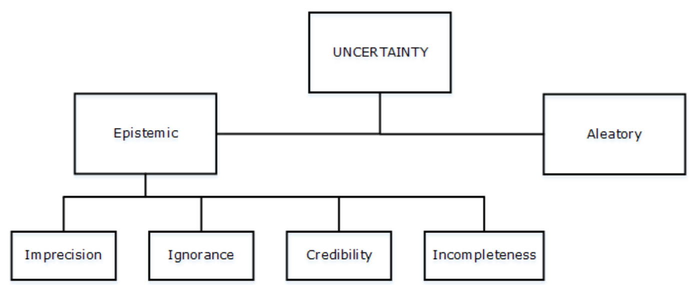

Figure 1. Taxonomy of uncertainty.

Source: Simon, et al., (2018).

Having established these fundamental elements, various methods and theories are discussed below for dealing with uncertainty in decision processes.

The modern interpretation of probability is based on the axiomatic approach developed by Andrei Kolmogorov in his 1933 book Grundbegriffe der Wahrscheinlichkeitsrechnung (Foundations of probability theory). Considering the set of all possible scenarios U (universal set), it is assumed the possibility of assigning to each element a probability function satisfying the properties presented in equation 1. Then, the probability of occurrence of an event E is given, if discrete events are considered, using equation 2, while if the event space is continuous, the probability function cannot be defined.

$$p(x) \in [0,1]\ \nabla x \in U\ ;\ \sum_{x \in U} p(x) = 1 \tag{1}$$

$$P(E) = \sum_{x \in E} p(x) \tag{2}$$

Since its formalization the probabilities have taken two different meanings. In the first place, there is an objective meaning in which it represents the frequency of occurrence of the analyzed events, while from a subjective point of view the probabilities represent the degree of belief about the truth of a proposition. The first approach is very useful for analyzing situations in which a large amount of data is generated on the phenomenon under study, such as the quality control of processes in industries. The decision-maker can estimate, by observing the frequency of occurrence of an event, the probability of its occurrence and act accordingly. This is the principle of statistical inference. On the other hand, as Simon, et al. (2018), establish, in the subjective approach the probability is obtained from the state of knowledge of an expert. Any evidence that changes the degree of the expert's belief must be considered when calculating the probability using Bayes' theorem. Probability assessment is assumed to be performed by a coherent expert where any coherent expert having the same state of knowledge would make the same assessment.

Then, in the case that the actions and states of nature present a discrete nature, these can usually be summarized in a matrix as shown in Table 1 where, as Taha (2017), suggests, the element a_i represents action i and element s represents the state of nature j. The payoff or result associated with action a_i and state s_j is $v(a_i, s_j)$. In decisions under risk, each state of nature s_j can be assigned a probability $p(s_j)$. In this context, if the decision maker manifests a rational behavior, the optimal action is given by the criterion of the expected value which that, depending on the case analyzed, is the one that maximizes the expected utility or minimizes the expected cost as shown in equation 3.

$$a^* = arg \left\{ \underset{i}{max/min} \left[\sum_{j=1}^{n} v(a_i, s_j) \cdot p(s_j) \right] \right\} \tag{3}$$

Table 1. Payoff matrix.

	s_1	s_2	…	s_n
a_1	$v(a_1,s_1)$	$v(a_1,s_2)$	…	$v(a_1,s_n)$
a_2	$v(a_2,s_1)$	$v(a_2,s_2)$	…	$v(a_2,s_n)$
…	…	…	…	…
a_m	$v(a_m,s_1)$	$v(a_m,s_2)$	…	$v(a_m,s_n)$

Source: Taha (2017).

It is common to estimate through the analysis of historical data that the probabilities of the analyzed phenomenon, although in some contexts these can be improved through experimentation. The probabilities estimated by the first way are known as ***a priori*** while in the second case they are known as ***a posteriori***. Hillier & Lieberman (2015), point out it is sometimes argued that these probability estimates are necessarily subjective and therefore cannot be relied upon, but they emphasize that although this argument has some validity, in many circumstances experience allows the development of reasonable estimates which provides a stronger foundation for making a good decision.

Decisions under complete ignorance are those in which there is no information regarding the possibility of occurrence of the states of nature, that is, it is difficult or impossible to assign probabilities. In this context decision-making is particularly difficult; so special criteria have been developed. The selection among these depends on the decision-maker's attitude to risk, and will be addressed below.

Laplace's criterion is based on the principle of insufficient reason. Since the probability distributions are not known, there is no reason to believe that the probabilities associated with the states of nature are different, therefore the alternatives are evaluated using the simplifying assumption that all states are equally probable that occur; that is, if the payoff $v(a_i, s_j)$ represents the profit, the best alternative is the one that obtains the best performance in equation 4 (Taha, 2017).

$$a^* = \underset{a_i}{max} \left\{ \frac{1}{n} \sum_{j=1}^{n} v(a_i, s_j) \right\} \tag{4}$$

The Wald criterion, or max-min strategy, consists of choosing the action a* that corresponds to making the best of the worst possible conditions (equation 5). This criterion reflects an attitude of extreme pessimism, because the decision maker operates as if, whatever action he chooses, the state of nature will occur (in terms of structural, political and economic conditions) that will provide him with the least payoff. Therefore, the decision maker protects himself from nature by trying to achieve the maximum of the minimum benefits. On the other hand, the max-max criterion (extremely optimistic perspective) considers that, whatever the action chosen, nature will be so benign as to grant maximum well-being (equation 6) (Bacci & Chiandotto, 2020).

$$a^* = \underset{i}{max}\left\{\underset{j}{min}\, v(a_i, s_j)\right\} \tag{5}$$

$$a^* = \underset{i}{max}\left\{\underset{j}{max}\, v(a_i, s_j)\right\} \tag{6}$$

The Hurwicz criterion represents a compromise between pessimistic and optimistic attitudes by introducing a variable $\alpha \in [0,1]$ that represents the degree of optimism of the decision maker (equation 7). It can be verified that if $\alpha = 0$, equation 7 reduces to the max-min criterion, while if $\alpha=1$ it would then be the max-max criterion. The degree of optimism (pessimism) can be adjusted by selecting a value of α between 0 and 1 and as Taha (2017), suggests without the strong feeling regarding extreme optimism or pessimism, $\alpha=0{,}5$ may be a fair choice.

$$a^* = arg\left\{\underset{i}{max}\left[(1-\alpha)\cdot \underset{j}{min}\, v(a_i, s_j) + \alpha \cdot \underset{j}{max}\, v(a_i, s_j)\right]\right\} \tag{7}$$

Finally, the Savage criterion proposes the minimization of the regret that the decision maker may suffer when choosing a strategy that would not be optimal. For this, a loss matrix is constructed by finding the difference between each element of the decision matrix and the maximum value of the row as shown in equation 8. Then the optimal action that minimizes the maximum regret is selected according to the equation 9. According to Carter, et al. (2019), this strategy is associated with insecure decision makers who are not primarily interested in making the biggest profits, but are more concerned with how disappointed they will be after the decision.

$$r(a_i, s_j) = \underset{i}{max}\, v(a_i, s_j) - v(a_i, s_j) \tag{8}$$

$$a^* = arg\left\{\underset{i}{min}\left[\underset{j}{max}\, v(a_i, s_j)\right]\right\} \tag{9}$$

The methods discussed thus until now are part of the basic operations research study program for the decision theory course. However, the main disadvantage of these approaches is that for their implementation it is required an amount of information that is often not available. An example would be the impossibility to know all possible future states of nature or the probabilities of those states. Without this knowledge, it is not possible to define a conditional probability of all outcomes or calculate their expected profits (Polasky, et al., 2011). Another common problem is the estimation of the payoff of implementing an action [$v(a_i, s_j)$] which can only be carried out in a vague or imprecise way, which introduces uncertainty in the result. In order to face these situations other theories which don't rely on probabilities have been developed and will be discuss next.

Fuzzy set theory has been developed for solving problems in which descriptions of activities and observations are imprecise, vague, and uncertain. The term "fuzzy" refers to the situation in which there are no well-defined boundaries of the set of activities or observations to which the descriptions apply. For example, one can easily assign a person seven feet tall to the "class of tall men", but it would be difficult to justify the inclusion or exclusion of a six-foot tall person to that class, because the term "tall" does not constitute a well-defined boundary. That's why these classes of objects cannot be well represented by classical set theory (Chen & Hwang, 1992).

According to Chaira (2019), a classical set is normally defined as a collection of objects or elements x in $X=(x_1, x_2, x_3, \ldots, x_n)$ that are finite. Most of our traditional tools, modeling, and methods are based on crisp set theory where elements are deterministic and precise. This means that the statement is either "true or false" and in mathematics it may be defined as either "0 or 1." In classical sets elements have a Boolean state of nature that means either belongs to the set or does not belong to the set and this belongingness is termed as "membership value" or the degree of belongingness. So, if an element in a set is present, then its membership value is "1" else its membership value is "0".

This values of membership can be generalized so that the values assigned to the elements of the universal set X fall within the specified range [0,1]. The assigned value indicates the degree of membership of the element in the considered set; higher values denote higher degrees of established membership. Such a function is called a membership function (MF) and the set defined by this function is called a fuzzy set. In essence, the membership function of a fuzzy set $\tilde{A}$, denoted by $\mu_{\tilde{A}}$ maps elements of the universal set X into real numbers in [0,1]; that is, $\mu_{\tilde{A}}: X \to [0,1]$. Therefore, in this case the fuzzy set $\tilde{A}$ is completely characterized by the set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ where the second component of this ordered pair declares the degree membership of the first component of fuzzy set A (Ebrahimnejad & Verdegay, 2018).

The interactions between different fuzzy sets are considered operations on fuzzy sets, but being different from crisp sets, operations on fuzzy sets are defined based on the membership function rather than the set itself. Klir & Yuan (1995), highlight that among the different types of fuzzy sets of special significance are those that are defined in the set of real numbers $\mathbb{R}$. These have a quantitative meaning and under certain conditions can be seen as fuzzy numbers which are essential to characterize states of fuzzy variables. Fuzzy numbers can take many different forms (Figure 2) allowing to model objects or events, which from a mathematical perspective could be in a possible range defined by quantitative limits being useful in the description of categories subject to uncertainty due to vagueness or imprecision.

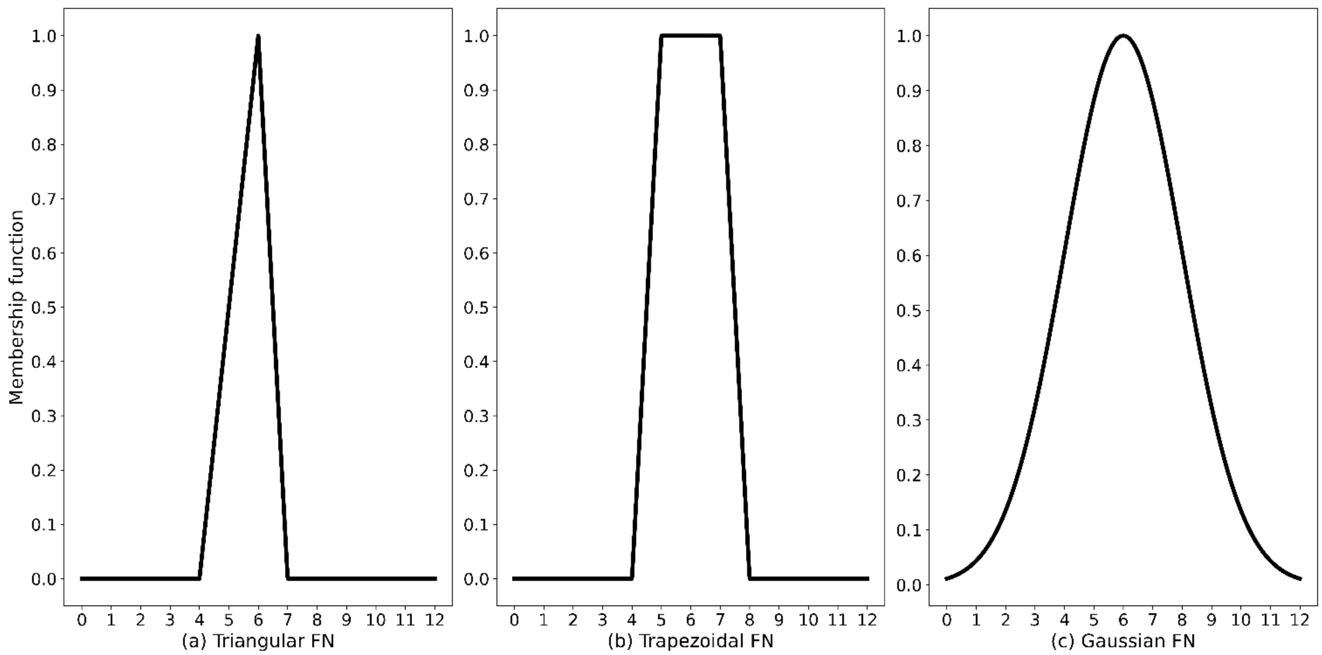

Figure 2. Examples of fuzzy numbers.

Taking this into account Dubois & Prade (1980), argue that a stochastic method such as statistical decision analysis does not measure imprecision in human behavior, instead this method is a way to model incomplete information about the environment external to human beings. The theory of fuzzy sets, on the other hand, allows modeling the uncertainty (or imprecision) caused by mental phenomena that are not random or stochastic in nature. Bellman & Zadeh (1970), were the first to discuss the applicability of fuzzy sets to strengthen decision-making through the use of verbal expressions which according to Espín Andrade, Fernández González & González Caballero (2014), constitutes the main advantage of its use. A preferential knowledge representation based on fuzzy logic gives the opportunity to use language as an element of communication and modeling in decision analysis, creating an explicit model of preferential knowledge; and subsequently use the inference capacity of the logical platform to propose decisions that better reflect the decision policy of the human agent.

Despite the initial skepticism of abandoning the theory of probability, the theory of fuzzy sets has maintained a continuous development allowing advances in decision making and many fields as can be seen in Bělohlávek, Dauben & Klir (2017). The use of linguistic variables has contributed to the popularity of this approach given its low computational cost and ease of understanding. This theory has arisen as a powerful tool to deal with the complexities that exist in real world problems however, in the face of new demands the theory of fuzzy sets has been extended, as well as other alternative tools have emerged.

Fuzzy set theory takes into account membership degree, and the nonmembership degree is the complement of the membership degree $[1-\mu_{\tilde{A}}(x)]$. However, in real life, this linguistic negation does not satisfy the logical negation because there may be some kind of hesitation while defining the membership function. Due to this reason, Atanassov (1986), suggested an intuitionistic fuzzy set (IFS) where the nonmembership degree is not equal to the complement of the membership degree but it has to be defined in order to take into account hesitation or lack of knowledge. So, compared to fuzzy set theory, IFS considers two uncertainties –membership and nonmembership degrees (Chaira, 2019).

As defined by Atanassov (1999), an intuitionistic fuzzy set (IFS) A in E is defined as an object of the following form $A=\{(x,\mu_A(x),\nu_A(x))|x\in E\}$ where the functions $\mu_A:E\rightarrow[0,1]$ and $\nu_A:E\rightarrow[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in E$ respectively, and for every $x\in E$ it is fulfilled that $0\leq\mu_A(x)+\nu_A(x)\leq 1$. The value of $\pi_A(x)=1-\mu_A(x)-\nu_A(x)$ is called the degree of non-determinacy (or uncertainty) of the element $x\in E$ to the intuitionistic fuzzy set. According to Pękala (2019), this value is a measure of lack of knowledge and is useful in important applications like when considering the distances, entropy and similarity for the intuitionistic fuzzy sets, being crucial in virtually all information processing tasks.

On the other hand, Smarandache (1998), extended intuitionistic fuzzy sets calling these new structures neutrosophic sets. The term neutrosophy derives from the French ***neutre*** which derives from the Latin ***neuter*** meaning neutral, and from the Greek ***sophia***, which means wisdom. Then, the term neutrosophic means knowledge of neutral thought. According to Smarandache, et al. (2019), neutrosophy leads to an entire family of novel mathematical theories with an overview of not only classical but also fuzzy counterparts. The reason is that a fuzzy set representing uncertainty exists in the attributes using single-valued membership. In this case, one cannot represent for example when win, loss, and draw match independently. To represent this, it is needed to characterize them lay in membership-values of truth, falsity, and indeterminacy. This makes it necessary to extend the fuzzy sets beyond acceptation and rejection regions using single-valued neutrosophic values. It contains truth, falsity, and indeterminacy membership values for any given attribute and the most interesting point is that all these three functions are completely independent, and one function is not affected by another. Then this new theory essentially studies the starting point, environment, and range of neutralities and their exchanges with ideational ranges.

Both, intuitionistic fuzzy sets and neutrosophic sets, generalize fuzzy sets and as these have already accomplished a great success in different applications it is expected that these extensions could be used as well in decision-making processes and other fields where inevitably it is needed to deal with impreciseness or vagueness. However fuzzy set theory and its extensions are not the only way to deal with uncertainty arisen due to epistemic uncertainty as will be discuss below.

The concept of rough set was originally introduced by Pawlak (1982), to model vagueness. In this point it is well known that fuzzy set theory developed by Zadeh (1965), can be used to do that however, the concepts of vagueness in these theories should be distinguished. Fuzzy set theory deals with gradualness of knowledge by using fuzzy membership, whereas rough set theory deals with granularity of knowledge (Figure 3) by using indiscernibility relation under background knowledge (Akama, Kudo & Murai, 2020). As pointed out by Pawlak & Skowron (2006), the rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, knowledge). For example, if objects are patients suffering from a certain disease, symptoms of the disease form information about patients. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. The indiscernibility relation generated in this way is the mathematical basis of rough set theory. This understanding of indiscernibility is related to the idea of Gottfried Wilhelm Leibniz that objects are indiscernible if and only if all available functionals take on identical values. However, in the rough set approach, indiscernibility is defined relative to a given set of functionals (attributes).

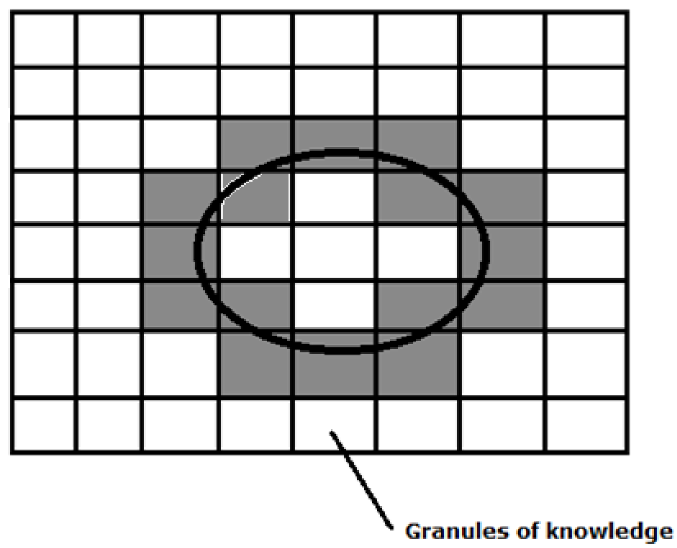

Figure 3. Notion of granules of knowledge in a rough set.

Mathematically, let $X\subseteq U$ and R be an equivalence relation. We say that X is R-definable if X is the union of some R-basic categories; otherwise X is R-undefinable. The R-definable sets are those subsets of the universe which can be exactly defined in the knowledge base K, whereas the R-undefinable sets cannot be defined in K. The R-definable sets are called R-exact sets, and R-undefinable sets are called R-inexact or R-rough. Set $X\subseteq U$ is called exact in K if there exists an equivalence relation $R\in IND(K)$ such that X is R-exact, and X is said to be rough in K if X is R-rough for any $R\in IND(K)$. Observe that rough sets can be also defined approximately by using two exact sets, referred as a lower and an upper approximation of the set. Suppose we are given knowledge base K=(U,R) . With each subset $X \subseteq U$ and an equivalence relation $R \in IND(K)$ we associate two subsets $\overline{R}X = \cup\{Y \in U/R : Y \subseteq X\}$ and $\underline{R}X = \cup\{Y \in U/R : Y \cap X \neq \emptyset\}$ called the R-lower approximation and the R-upper approximation of X, respectively. A pair $(\overline{R}(X), \underline{R}(X))$ is called the rough set with respect to X (Akama, et al., 2020).

Applications of the RST are varied, such as data analysis and reduction, generation of decision rules, image processing, pattern recognition, knowledge discovery, knowledge representation, and concept naming. There are several kinds of problems that can be solved using the rough set approach, such as: (1) description of a set of objects in terms of the attribute values; (2) dependencies between attributes; (3) reduction of attributes; (4) significance of attributes; and (5) generation of decision rules. According to Tay & Shen. (2002), the rough set approach has several advantages: (1) it can perform the analysis straightforwardly using the original data only and does not need any external information such as probability in statistics or grade of membership in the fuzzy set theory, (2) it is suitable for analyzing not only quantitative attributes but also qualitative ones; (3) it can discover important facts hidden in data and expresses them in the natural language of decision rules; (4) the set of decision rules gives a generalized description of the knowledge contained in the information tables; and (5) the results of the rough sets analysis are easy to understand by the natural language (Tzeng & Huang, 2011).

Grey system on the other hand deals with uncertainty due to incomplete or unknown information. It is a theory introduced by Deng (1982), which is effective in the study of problems involving small samples and poor information. These kind of problems cannot be handled effectively by the theories discussed before and as they arise frequently in ecological, social, economic, biological and many other systems, since its conception grey system theory has generated a great interest. The term grey refers to the degree of known information about the system: black represents an unknown information, white completely known information, and "grey" is that information which is partially known and partially unknown. As pointed out by Liu & Lin (2006), during the initial establishment and the consequent development of the theory, many important axioms have been deriving the principles shown in Table 2.

Table 2. Fundamental Principles of Grey Systems.

Number	Principle	Interpretation
1	Principle of Informational Differences	"Difference" implies the existence of information. Each piece of information must carry some kind of difference
2	Principle of Non-Uniqueness	The solution to any problem with incomplete and nondeterministic information is not unique
3	Principle of Minimal Information	One characteristic of grey systems theory is that it makes the most and best use of the available "minimal amount of information"
4	Principle of Recognition Base	Information is the foundation on which people recognize and understand (nature).
5	Principle of New Information Priority	The function of new pieces of information is greater than that of old pieces of information
6	Principle of Absoluteness of Greyness	"Incompleteness" of information is absolute

Source: Liu & Lin (2006).

The concept of grey hazy sets is the set-theoretic basis of grey systems theory, which includes four types of numbers: grey numbers, number-covered set, whitened numbers, and the only potential true numbers. A grey number is an uncertain quantity, a quantitative expression of the connotation of the matter of concern, and a basic element of grey mathematics. Because of the relativity of the matter's cognitive connotation and the characteristic of poor information, one can only obtain a covered set of the connotation, whose quantitative expression is the number-covered set of the grey number. A whitened number can be any particular value within the number-covered set; it represents an approximation of the true value of the grey number. The true value of the grey number has to be within the number-covered set. Because of the uniqueness of connotations, the true value of any grey number exists uniquely, known as the only potential true number. When the connotation of a matter is completely known, the grey uncertain number turns into a real number. That is why grey mathematics takes the same forms of operation of the conventional mathematics. But, at the same time, grey mathematics possesses its own additional rules and particular forms of operation. There are four main forms of operation in grey mathematics: grey operations, covered operations, whitened operations, and operations of only potential true numbers. Here, operations between grey numbers are known as grey number operations, those between number-covered sets as number-covered operations, those between only potential true numbers as potential true operations, and those between whitened values as whitened operations (Li & Lin, 2014).

Different approaches for the use of grey system theory can be consulted in Liu & Lin (2010), while applications in data processing, modeling, prediction, control and decision making may be found in literature. However, in spite of very important results accomplish by the use of the theory its theoretical framework needs further completion.

According to Li & Lin (2014), due to the complexity of the external environment and the inherent limitation of human ability to acquire information, various types of uncertain phenomena, different of those of stochastic problems, are discovered in the human efforts of understanding and changing the world. These discoveries led to three additional methodologies of uncertain systems developed to deal with the corresponding problems of uncertainty. These methodologies are respectively the fuzzy systems theory, initiated by Zadeh, rough set theory, established by Pawlak, and grey systems theory, proposed by Deng (1982). Presently, together with probability they are jointly known as four systems scientific methods developed for dealing with uncertainties.

After the introduction of these theories several researchers have extended or combined them in order to take advantage of their particular strengths. An example can be rough fuzzy sets and fuzzy rough sets as debated in Dubois & Prade (1990). A comprehensive discussion is out of the scope of this paper although in Table 3 is presented a short historical review.

Table 3. Extensions and combination of approaches to deal with uncertainty.

Year	Advance
1967	Goguen presents L-fuzzy sets
1971	Zadeh introduces type-2 fuzzy set
1975	Zadeh presents the definition of type-n fuzzy set
1975	Sambuc proposes the concept of an interval-valued fuzzy set under the name of H-Flou Sets. Zadeh suggests the same notion of interval-valued fuzzy set of H-Flou Sets. Zadeh suggests the same notion of interval-valued fuzzy set as a particular case of type-2 fuzzy sets
1976	Grattan-Guinness presents the notion of set-valued fuzzy set as well as some operations based on previous developments for many-valued algebras
1982	Pawlak introduce the concept of rough set
1983	Atanassov presents the definition of intuitionistic fuzzy set
1986	Yager gives the idea of fuzzy multiset
1989	Atanassov and Gargov present the notion of interval-valued Atanassov intuitionistic fuzzy set
1989	Grey sets are defined by Deng
1990	Dubois and Prade present the notion of fuzzy rough sets
1993	Gau and Buehrer define the concept of vague set
1996	Zhang presents the definition of bipolar valued fuzzy set
1998	Pedrycz introduces the notion of shadow set
2000	Liang and Mendel introduce the idea of interval type-2 fuzzy set
2000	Lee introduces a new concept with the name of bipolar valued fuzzy set
2001	Maji, Biswas and Roy introduce the notion of fuzzy soft set
2002	Smaradache introduces the concept of neutrosophic set
2002	Kandel introduces the concept of complex fuzzy set
2010	Torra introduces the notion of hesitant fuzzy set
2013	Yager gives the idea of pythagorean fuzzy set
2014	Bedregal et al. introduce the notion of typical hesitant fuzzy set
2014	Mesiarova-Zemankova et al. present the concept of m-polar-valued fuzzy set

Source: Pękala (2019).

CONCLUSIONS

The conception of uncertainty goes beyond randomness, and depending the source of it several methodologies can help in the decision making process. Fuzzy mathematics and its extensions abandon Bolean logic to consider all possible outcomes when objects are not sharply defined and in order to do that they rely on experience and experts' knowledge. The approach of rough sets to model vagueness is different than fuzzy mathematics paying more attention to indiscernibility of the available information. Grey system theory on the other hand deals with poor information and small samples. Due to the necessity of increase the robustness of decision making through the consideration of uncertainty these methodologies have generated great interest in the scientific community combining them in order to tackle problems with an increasing complexity. However, as any other mathematical tool its use depends on the system analyzed and the goals the analyst is aiming at.

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